Differential Geometry IV: General relativity

8.1 (a) Let $\mathcal{D}$ be the domain in the Einstein cylinder $(\mathbb{R}\times\mathbb{S}^3, g_E)$, $g_E = -dt^2 + g_{\mathbb{S}^3}$, which is the image of Minkowski spacetime $(\mathbb{R}^{3+1}, \eta)$ under the conformal map $(u, v) \to (\mathrm{Arctan}(u), \mathrm{Arctan}(v))$ that we saw in class. Let p be a point on future null infinity $\mathcal{I}^+ \subset \partial\mathcal{D}$ and consider the set $C^-(p) \cap \mathcal{D}$ of past null geodesics emanating from p restricted to $\mathcal{D}$. How does this set look like in the standard Cartesian coordinates of $\mathbb{R}^{3+1}$? Deduce that every pair of null geodesics of $(\mathbb{R}^{3+1}, \eta)$ asymptoting in the future to the same point on $\mathcal{I}^+$ have to asymptote in the past to the same point on $\mathcal{I}^-$.

(b) Let $(\mathcal{M}^{3+1}, g)$ be a spherically symmetric spacetime without axis, i.e. $\mathcal{M}^{3+1} = \mathcal{U}^{1+1} \times \mathbb{S}^2$ and, in any local coordinates (x^1, x^2) on $\mathcal{U}$,

$$g = \tilde{g}_{AB} dx^A dx^B + r^2 g_{\mathbb{S}^2},$$

where $\tilde{g}$ is a Lorentzian metric on $\mathcal{U}$ and $r : \mathcal{U} \to (0, +\infty)$ is a smooth function. The spacetime $(\mathcal{U}, \tilde{g})$ is known as the *Penrose diagram* of $(\mathcal{M}, g)$ and can be formally thought of as the projection $\mathcal{U} = \mathcal{M}/SO(3)$. Show that the image $\tilde{\gamma}$ in $\mathcal{U}$ of a causal curve γ in $\mathcal{M}$ is again a causal curve (hence, Penrose diagrams are useful 2-dimensional tools to read-off the causal structure of a 4 dimensional spacetime). In which case is the projection of a null curve in $\mathcal{M}$ again a null curve in $\mathcal{U}$?

8.2 (a) Show that the Schwarzschild metric

$$g_M = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{1}$$

on $\mathcal{M} = \mathbb{R}_t \times (2M, +\infty)_r \times \mathbb{S}^2$ is indeed a solution of the vacuum Einstein equations.

(b) Show that $(\mathcal{M}, g_M)$ embeds isometrically into $\tilde{\mathcal{M}} = \mathbb{R}_{t^*} \times (0, +\infty)_r \times \mathbb{S}^2$ with

$$g_{\tilde{\mathcal{M}}} = -\Big(1 - \frac{2M}{r}\Big)(dt^*)^2 + \frac{4M}{r} dt^* dr + \Big(1 + \frac{2M}{r}\Big)dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)$$

(*Hint: Use the coordinate transformation $t^* = t + f(r)$ for a suitable $f(r)$.*) Show that, in the extended spacetime, the region $\{r \leqslant 2M\}$ corresponds to a *black hole*, that is to say, no future directed causal curve starting from $\{r \leqslant 2M\}$ can end up in the asymptotically region $\{r \gg 1\}$.

8.3 (a) Let $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a geodesic in the Schwarzschild (exterior) spacetime $(\mathcal{M}, g_M)$. Show that the geodesic equation takes the form

$$\begin{aligned}
\frac{d}{ds}\Big(\big(1 - \frac{2M}{r}\big)\dot{t}\Big) &= 0,\\
\frac{d}{ds}\Big(\big(1 - \frac{2M}{r}\big)^{-1}\dot{r}\Big) &= \frac{1}{2}\Big(-\frac{2M}{r^2}\dot{t}^2 - \big(1 - \frac{2M}{r}\big)^{-2}\frac{2M}{r^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\Big),\\
\frac{d}{ds}\Big(r^2\dot{\theta}\Big) &= \frac{1}{2}r^2\sin\theta\cos\theta\dot{\phi}^2,
\end{aligned}$$

$$\frac{d}{ds}\Big(r^2\sin^2\theta\dot{\phi}\Big)=0.$$

Deduce that one can without loss of generality one can consider geodesics lying in the equatorial plane $\theta=\frac{\pi}{2}$ (by possibly rotating the coordinate system (θ,ϕ) on $\mathbb{S}^2$.) Note that, in this case, the first and fourth of the equations above reduce to the statement that the energy E and angular momentum L of a geodesic are constant (i.e. are constants of motion for the geodesic flow).

(b) Show that there exist "trapped" null geodesics orbitting the black hole (i.e. null geodesics that never approach $r=2M$ or $r=\infty$) (*Hint: For an appropriately chosen value of* $r=r_0>2M$, *show that there exist null geodesics with* $r(s)=r_0$ *for all* s.. Contrast this with the situation on Minkowski spacetime.
Remark. The region traced out by trapped null geodesics consists the so-called *photon sphere* of a black hole.

(c) Show that, for any $\mu>0$, there exist timelike geodesics γ in the Schwarzschild spacetime with $g(\dot{\gamma},\dot{\gamma})=-\mu$ which are trapped. (*Hint: It might be convenient, instead of working with the second order equations, to use the invariants of the geodesic flow and obtain a relation for* $\dot{r}$ *and observe that, for an appropriate choice of* E,L *and* $r(0)$, $r(s)$ *cannot escape a bounded interval in* r.)
Remark. These timelike orbits correspond to massive objects (e.g. planets) moving under the influence of gravity in Schwarzschild spacetime. Unlike the trapped null geodesics, these orbits are *stable*, namely they remain trapped even under small perturbations of the initial condition $r(0)$ and the conserved quantities E,L,μ (were these trapped orbits not stable, earth would plunge in the sun under small perturbations of its orbit).